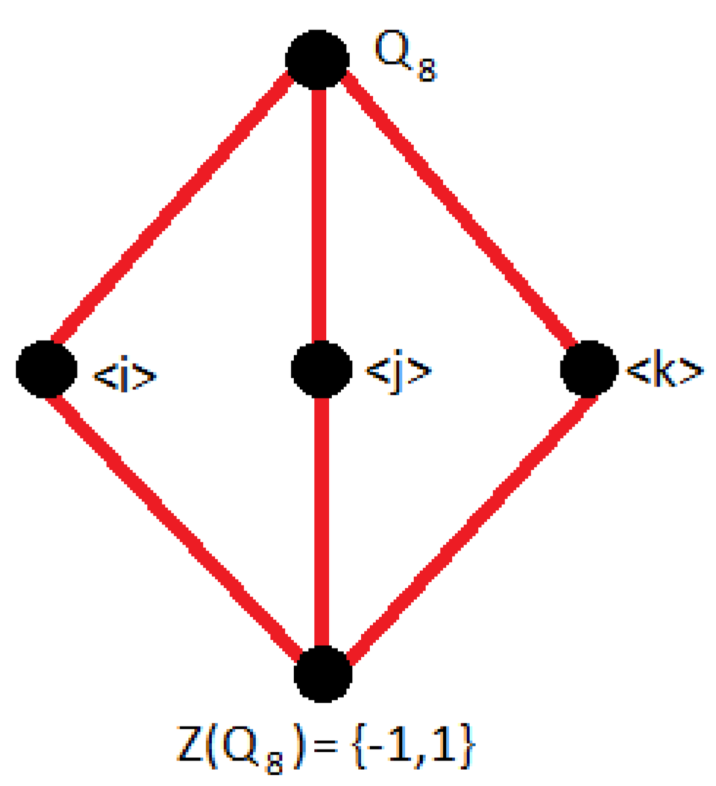
Chermak-Delgado Simple Groups

Ryan McCulloch, School of Arts & Sciences, University of Bridgeport

Introduction

In mathematics, a group is a set of elements equipped with a binary operation that satisfies the axioms of identity, inverses, and associativity. A subgroup of a group is a subset which is itself a group under the same binary operation. Subgroups of a group can be displayed in a subgroup lattice, which is a diagram where lines are drawn to illustrate set containment, and which as an algebraic structure satisfies the conditions of join and meet. A normal subgroup of a group is a subgroup which is invariant under the action of conjugation. The normal subgroup lattice of a group has nice properties. For example, in the normal subgroup lattice, the join of two subgroups is simply the product of those subgroups. In 1989, A. Chermak and A. Delgado[2], discovered a new subgroup lattice for finite groups, which has since been called the Chermak-Delgado lattice. This lattice consists of those subgroups which have maximal Chermak-Delgado measure. The CD-measure is defined as the product of the order of the subgroup and the order of its centralizer subgroup. The Chermak-Delgado lattice shares many nice properties in common with the normal subgroup lattice, but it has the unique property that it is self-dual, which means that if you flip it upside down, you get the exact same diagram. Below is the CD lattice for the quaternions group, Q_8.

The CD Lattice of Q_8

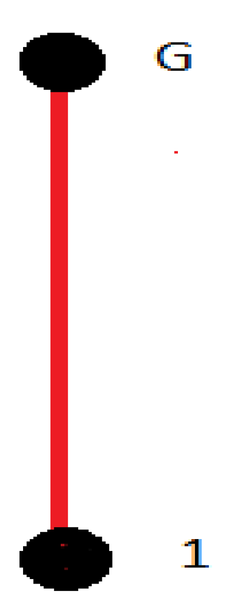


CD-simple Groups

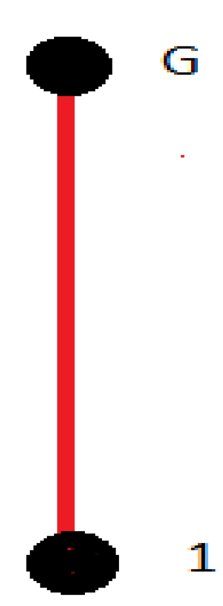
A simple group is a group, G, whose only normal subgroups are 1 and G. To describe simple groups, I use the following analogy to chemistry: just as the elements are the building blocks of matter, the finite simple groups are the building blocks of all finite groups. The classification of finite simple groups was a massive project, consisting of hundreds of papers from many mathematicians, and was finally completed in the early 2000's. So what's next?

As mentioned in the introduction, the Chermak-Delgado lattice shares many similar properties with the normal subgroup lattice, however it has the additional property of being self-dual. A simple group has its normal subgroup lattice consisting of only 1 and G. So I have defined a CD-simple group to be a group whose Chermak-Delgado lattice consists of only 1 and G. See the diagrams below.

The Normal Subgroup Lattice of a Simple Group



The CD Lattice of a CD-simple Group



Notes:

- Every simple group is CD-simple, but the converse is not true. So the notion of “CD-simple” is a generalization of the notion of “simple”.
- There are examples of solvable CD-simple groups. (A solvable group, without getting into too many details, is a group whose “building blocks” consist of groups of prime order). Other than the simple groups of prime order, simple groups are never solvable.
- Thus, a good starting point in classifying the CD-simple groups is to classify the solvable ones.
- This is my ultimate goal.

My Results

In my paper[3], the first theorem is the most elegant, and provides additional motivation for studying CD-simple groups (in the hopes of attracting more researchers to the subject). An indecomposable group is a group which cannot be decomposed into a direct product of nontrivial subgroups. I define Property A to be the property that for each nontrivial abelian normal subgroup, A, of a group G, the quotient group $G/C_G(A)$ embeds into Aut(A) as a subgroup of order larger than A. I then show that the indecomposable groups having Property A are precisely the CD-simple groups. The implication of this theorem is that if an indecomposable group has the property that for **EVERY** subgroup H, the index $|G : C_G(H)| > |H|$, then it suffices to verify this property only for the abelian normal subgroups, A, of G. Abelian normal subgroups are much easier to locate, so this illustrates the power of using CD-measure arguments in finite group theory. In fact, the original results of A. Chermak and A. Delgado were used to refine and provide more elegant argument in the massive library of theorems that comprise the classification of finite simple groups.

Further Results and Conclusions

The second half of my paper is more technical, and works towards to worthwhile goal of classifying all of the solvable CD-simple groups. I achieve this classification for groups of order pq^a where p and q are primes. The classification is in terms of the action of $G/C_G(N)$ on a minimal normal subgroup N, where N can be viewed as a vector space. The techniques are rooted in representation theory, but all arguments are “character free”. Finding elegant and “character free” proofs are a prize in finite group theory. The next step would be a classification of groups of order p^aq^b. The famous Burnside p^aq^b Theorem[1] of the early 1900's shows that all groups of such order are solvable, and it is one of the first applications of character theory. Since then, “character free” proofs of Burnside's Theorem have been found. In conclusion, Chermak-Delgado measure arguments are a new tool to give elementary and elegant proofs in finite group theory. CD-simple groups are a natural generalization of simple groups, and work should be done to classify them.

References

[1] W. Burnside, *On Groups of Order p^aq^b*, Proc. London Math. Soc. **2** (1904), 388-392

[2] A. Chermak & A. Delgado, *A Measuring Argument for Finite Groups*, Proc. AMS **107** (1989), 907-914

[3] Ryan McCulloch, *Chermak-Delgado Simple Groups*, Communications in Algebra (to appear)